

# INFINITELY MANY EXCLUDED MINORS FOR FRAME MATROIDS AND FOR LIFTED-GRAPHIC MATROIDS

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**ABSTRACT.** We present infinite sequences of excluded minors for both the class of lifted-graphic matroids and the class of frame matroids.

## 1. INTRODUCTION

A matroid  $M$  is a *frame matroid* if there is a matroid  $M'$  with a basis  $V$  such that  $M = M' \setminus V$  and, for each  $e \in E(M)$ , the unique circuit in  $V \cup \{e\}$  has size at most 3. A matroid  $M$  is *lifted-graphic* if there is a matroid  $M'$  with  $E(M') = E(M) \cup \{e\}$  such that  $M' \setminus e = M$  and  $M'/e$  is graphic. The classes of lifted-graphic matroids and frame matroids were introduced by Zaslazsky [6] who proved that they are minor-closed.

We dispel the widespread belief that these classes would likely have only finitely many excluded minors.

**Theorem 1.1.** *There exist infinitely many pairwise non-isomorphic excluded minors for the class of frame matroids.*

**Theorem 1.2.** *There exist infinitely many pairwise non-isomorphic excluded minors for the class of lifted-graphic matroids.*

Our excluded-minors are based on constructions introduced by Chen and Whittle [2].

The existence of an infinite set of excluded minors does not necessarily prevent us from describing a class explicitly; see, for example, Bonin's excluded minor characterization for the class of lattice-path matroids [1]. The following conjectures would be a natural step towards finding such characterizations.

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**Conjecture 1.3.** *There exists an integer  $k$  such that all excluded minors for the class of lifted-graphic matroids have branch-width at most  $k$ .*

**Conjecture 1.4.** *There exists an integer  $k$  such that all excluded minors for the class of frame matroids have branch-width at most  $k$ .*

The class of quasi-graphic matroids, introduced in [3], contains both the lifted-graphic matroids and the frame matroids. In contrast to Theorems 1.1 and 1.2 we remain confident that the class of quasi-graphic matroids admits a finite excluded-minor characterization.

**Conjecture 1.5.** *There are, up to isomorphism, only finitely many excluded-minors for the class of quasi-graphic matroids.*

## 2. PRELIMINARIES

We assume that the reader is familiar with matroid theory and we follow the terminology of Oxley [4].

Recall that a *circuit-hyperplane* of a matroid  $M$  is a set  $C$  that is both a circuit and a hyperplane and that we can obtain a new matroid  $M'$  by relaxing a circuit-hyperplane  $C$  of  $M$ ; see [4, Proposition 1.5.14]. More specifically,  $\mathcal{B}(M') = \mathcal{B}(M) \cup \{C\}$  where  $\mathcal{B}(M)$  is the set of bases of  $M$ .

The reverse operation was introduced by Chen and Whittle [2]. A *free basis* of a matroid  $M$  is a basis  $B$  such that  $B \cup \{e\}$  is a circuit for each  $e \in E(M) - B$ . If  $B$  is a free basis of  $M$  then  $(E(M), \mathcal{B}(M) - \{B\})$  is a matroid (see [2]); we say that  $(E(M), \mathcal{B}(M) - \{B\})$  is obtained by *tightening*  $B$ .

Let  $G$  be a graph. For  $v \in V(G)$  we let  $\delta_G(v)$  denote the set of edges incident with  $v$ . For any  $U \subseteq V(G)$  and  $F \subseteq E(G)$ , let  $G[U]$  be the induced subgraph of  $G$  defined on  $U$ , and let  $G[F]$  be the subgraph of  $G$  with  $F$  as its edge set and without isolated vertices. A *cycle* of a graph is a connected 2-regular subgraph.

We assume that the reader is familiar with bias graphs; see Zaslavsky [5]. Let  $(G, \mathcal{B})$  be a bias graph. The cycles in  $\mathcal{B}$  are called *balanced* and a subgraph  $H$  of  $G$  is *balanced* if each of the cycles in  $H$  is balanced. A set  $F \subseteq E(G)$  is *balancing* if  $G \setminus F$  is balanced.

One well-known way to construct a bias graph is via a group-labelled graph. Here we use only the group of integers under addition, which we denote by  $\mathbb{Z}$ , and the group of non-zero real numbers under multiplication, which we denote by  $\mathbb{R}^\times$ . For an abelian group  $\Gamma$ , a  $\Gamma$ -labelled graph is a pair  $(\vec{G}, \gamma)$  where  $\vec{G}$  is an oriented graph and  $\gamma : E(\vec{G}) \rightarrow \Gamma$ . Let  $(\vec{G}, \gamma)$  be a  $\Gamma$ -labelled graph and let  $G$  be the undirected graph

underlying  $\vec{G}$ . A cycle  $C$  of  $G$  is *balanced* if the group-product of the labels on “clockwise” oriented edges is equal to the group-product of the labels on “counter-clockwise” oriented edges; this is independent of the direction on  $C$  we choose as clockwise. If  $\mathcal{B}$  is the set of balanced cycles of  $G$  then  $(G, \mathcal{B})$  is a biased graph.

For this paper it is more convenient to use the bias graph definition of frame matroids; see Zaslavsky [6]. Let  $(G, \mathcal{B})$  be a bias graph. We define  $FM(G, \mathcal{B})$  to be the matroid with ground set  $E(G)$  such that  $I \subseteq E(G)$  is independent if and only if  $G[I]$  has no balanced cycles and for each component  $H$  of  $G[I]$  we have  $|E(H)| \leq |V(H)|$ . Henceforth we will call a matroid  $M$  a *frame matroid* if and only if  $M = FM(G, \mathcal{B})$  for some bias graph  $(G, \mathcal{B})$ ; Zaslavsky [7] showed that this new definition is equivalent to the definition stated in the introduction.

Let  $M$  be a matroid. If  $(G, \mathcal{B})$  is a biased graph such that  $M = FM(G, \mathcal{B})$ , then  $\mathcal{B}$  is implicitly determined by  $G$  (and  $M$ ). Hence we refer to the graph  $G$ , itself, as a *frame representation* of  $M$ , and given a frame representation of a matroid we will refer to its cycles as balanced or non-balanced accordingly.

As with frame matroids, it is also more convenient to use the bias graph definition of lifted-graphic matroids; see Zaslavsky [6]. We define  $LM(G, \mathcal{B})$  to be the matroid with ground set  $E(G)$  such that  $I \subseteq E(G)$  is independent if and only if  $G[I]$  has at most one cycle and, should it exist, that cycle is non-balanced. Henceforth we will call a matroid  $M$  a *lifted-graphic matroid* if and only if  $M = LM(G, \mathcal{B})$  for some bias graph  $(G, \mathcal{B})$ ; Zaslavsky [8] showed that this new definition is equivalent to the earlier definition stated in the introduction.

Let  $M$  be a matroid. If  $(G, \mathcal{B})$  is a biased graph such that  $M = LM(G, \mathcal{B})$ , then  $\mathcal{B}$  is implicitly determined by  $G$  (and  $M$ ). Hence we refer to the graph  $G$ , itself, as a *lifted-graphic representation* of  $M$ , and given a lifted-graphic representation of a matroid we will refer to its cycles as balanced or non-balanced accordingly.

A cocircuit  $C^*$  of a matroid  $M$  is *non-separating* if  $M \setminus C^*$  is connected. If  $C^*$  is a non-separating cocircuit of a matroid  $M$  and  $M = FM(G, \mathcal{B})$  or  $M = LM(G, \mathcal{B})$ , then either  $C^*$  is a balancing set of  $(G, \mathcal{B})$  or  $C^* = \delta_G(v)$  for some vertex  $v \in V(G)$ .

### 3. FRAME MATROIDS

In this section we prove Theorem 1.1. Let  $k \geq 7$  be an odd integer. (The condition that  $k \geq 7$  is to simplify the proof;  $k \geq 3$  suffices.) Let  $(\vec{G}_k, \gamma)$  be the  $\mathbb{R}^\times$ -labelled graph defined in Figure 1 and let  $G_k$  denote

its underlying undirected graph. Let  $\mathcal{B}$  denote the balanced cycles of  $(\vec{G}_k, \gamma)$  and let  $N_k = FM(G_k, \mathcal{B})$ .

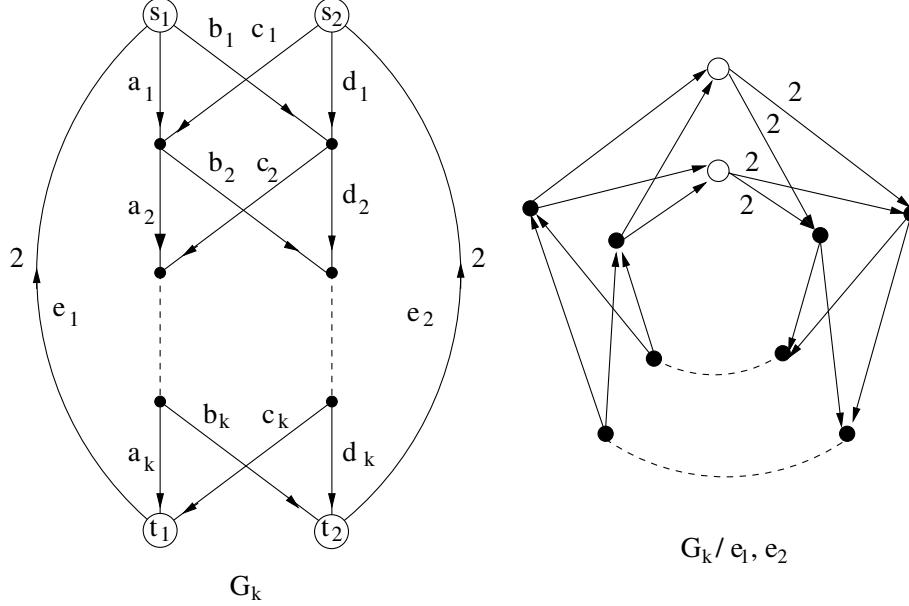


FIGURE 1. The graphs  $G_k$  and  $G_k/\{e_1, e_2\}$  with their  $\mathbb{R}^\times$ -labellings; unlabelled edges have group-label 1.

Let  $P = \{a_1, \dots, a_k\} \cup \{d_1, \dots, d_k\}$  and  $Q = \{b_1, \dots, b_k\} \cup \{c_1, \dots, c_k\}$ . Note that  $P \cup \{e_1, e_2\}$  and  $Q \cup \{e_1, e_2\}$  are free bases of  $N_k$ ; let  $M_k^F$  be the matroid obtained from  $N_k$  by tightening  $P \cup \{e_1, e_2\}$  and  $Q \cup \{e_1, e_2\}$ . Thus  $P \cup \{e_1, e_2\}$  and  $Q \cup \{e_1, e_2\}$  are circuits of  $M_k^F$ .

We will prove that  $M_k^F/\{e_1, e_2\}$  is an excluded minor. We start with the easier task of showing that proper minors of  $M_k^F/\{e_1, e_2\}$  are frame matroids.

**Lemma 3.1.** *For each  $e \in P \cup Q$ , both  $M_k^F/e$  and  $M_k^F \setminus e$  are frame matroids.*

*Proof.* Let  $M_P$  and  $M_Q$  denote the matroids obtained from  $M_k^F$  by relaxing the circuit hyperplanes  $P \cup \{e_1, e_2\}$  and  $Q \cup \{e_1, e_2\}$  respectively. For each  $e \in P$ , we have  $M_k^F \setminus e = M_P \setminus e$  and  $M_k^F/e = M_Q/e$ . Similarly, for each  $e \in Q$ , we have  $M_k^F/e = M_P/e$  and  $M_k^F \setminus e = M_Q \setminus e$ . So it suffices to prove that  $M_P$  and  $M_Q$  are frame matroids.

Note that  $M_P$  and  $M_Q$  are obtained from  $N_k$  by tightening the free bases  $Q \cup \{e_1, e_2\}$  and  $P \cup \{e_1, e_2\}$  respectively. Since  $G_k$  is a frame representation of  $N_k$  and  $G_k[Q \cup \{e_1, e_2\}]$  is a cycle in  $G_k$ , we have that  $G_k$  is a frame representation of  $M_P$ ; so  $M_P$  is indeed a frame matroid.

Let  $G'_k$  be the graph obtained from  $G_n \setminus \{e_1, e_2\}$  by adding  $e_1$  connecting  $s_1$  to  $t_2$  and adding  $e_2$  connecting  $s_2$  to  $t_1$ . It is straightforward to verify that  $G'_k$  is a frame representation of  $N_k$  (since  $\{e_1, e_2\}$  is a series pair in  $N_k$ ). Finally, since  $G'_k[P \cup \{e_1, e_2\}]$  is a cycle in  $G'_k$ , we have that  $G'_k$  is a frame representation of  $M_Q$ ; so  $M_Q$  is indeed a frame matroid.  $\square$

Now it remains to show that  $M_k^F / \{e_1, e_2\}$  itself is not a frame matroid.

**Lemma 3.2.**  $M_k^F / \{e_1, e_2\}$  is not a frame matroid.

*Proof.* Assume to the contrary that  $H$  is a frame representation of  $M_k^F / \{e_1, e_2\}$ . Let  $C_i = \{a_i, b_i, c_i, d_i\}$  for each  $i \in \{1, \dots, k\}$  and let  $G = G_k / \{e_1, e_2\}$ . Figure 1 depicts  $G$  with a group labelling encoding the balanced cycles with respect to  $N_k / \{e_1, e_2\}$ . From this group labelled graph we see that:

- (i) each cocircuit in  $N_k / \{e_1, e_2\}$  (and hence also in  $M_k^F / \{e_1, e_2\}$ ) has size at least 4,
- (ii) for each 4-element cycle  $C$  of  $G$ , the set  $E(C)$  is a circuit in  $N_k / \{e_1, e_2\}$  and, hence, also in  $M_k^F / \{e_1, e_2\}$ ,
- (iii) for each  $i \in \{1, \dots, k\}$ , the set  $C_i$  is a non-separating cocircuit in  $N_k / \{e_1, e_2\}$  and, hence, also in  $M_k^F / \{e_1, e_2\}$ , and
- (iv) for each  $v \in V(G)$ , the set  $\delta_G(v)$  is a 4-element non-separating cocircuit in  $N_k / \{e_1, e_2\}$  and, hence, also in  $M_k^F / \{e_1, e_2\}$ .

**3.2.1.**  $H$  is a simple connected 4-regular graph.

*Subproof.* Since  $N_k / \{e_1, e_2\}$  is connected, so is  $H$ . Since  $|E(H)| = |P \cup Q| = 2k = 2|V(H)|$ , we see that  $H$  has average degree 4. It follows from (i) that  $H$  is 4-regular. It remains to show that  $G$  is simple; suppose otherwise and let  $C$  be a cycle of length at most 2. At least  $3k - 6$  of the non-separating cocircuits described in (iii) and (iv) are disjoint from  $E(C)$ . Since  $k \geq 7$  we have  $3k - 6 > |V(H)|$  and hence one of these non-separating cocircuits is balancing. But then  $C$  is a circuit of  $M_k^F / \{e_1, e_2\}$  which is clearly absurd.  $\square$

For each 4-cycle  $C$  of  $G$ , the set  $E(C)$  is a circuit of  $M_k^F / \{e_1, e_2\}$  and, since  $G$  is simple,  $C$  is a cycle of  $H$ . Using this together with the fact that  $H$  is 4-regular it is routine to show that  $H$  is isomorphic to  $G$  and, moreover, that

- (a) there is an isomorphism that fixes  $C_1, \dots, C_k$  set-wise, and
- (b) for each  $i \in \{1, \dots, k\}$ , the sets  $\{a_i, d_i\}$  and  $\{b_i, c_i\}$  are matchings in  $H$ .

Now, since  $k$  is odd, one of  $H[P]$  and  $H[Q]$  is a cycle while the other is the union of two vertex-disjoint cycles. However  $P$  and  $Q$  are both

circuits in  $M_k^F/\{e_1, e_2\}$  which contradicts the fact that  $H$  is a frame representation.  $\square$

#### 4. LIFTED-GRAPHIC MATROIDS

In this section we prove Theorem 1.2. Let  $k \geq 3$  be an odd integer. Let  $(\vec{G}_k, \gamma)$  be the  $\mathbb{Z}$ -labelled graph defined in Figure 2 and let  $G_k$  denote its underlying undirected graph. Let  $\mathcal{B}$  denote the balanced cycles of  $(\vec{G}_k, \gamma)$  and let  $N_k = LM(G_k, \mathcal{B})$ .

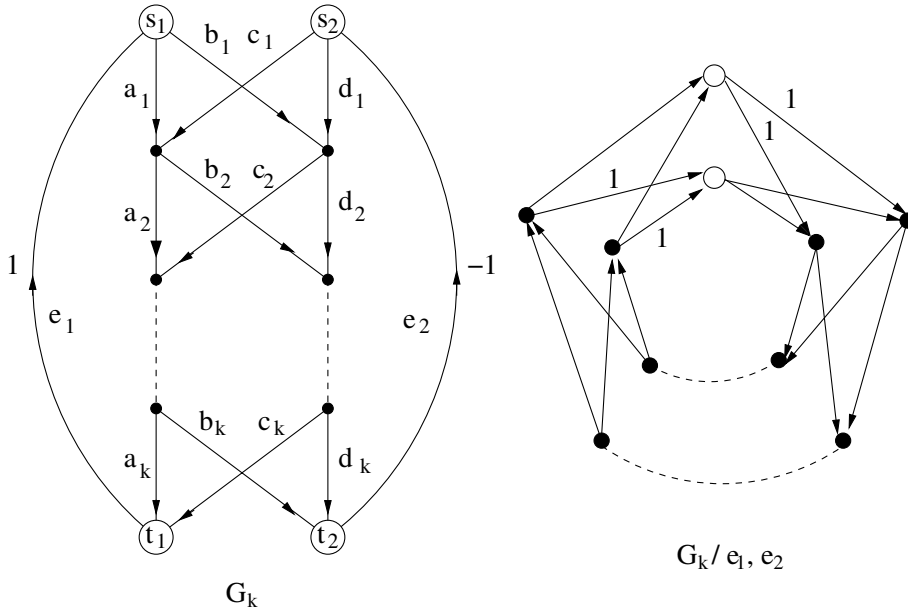


FIGURE 2. The graphs  $G_k$  and  $G_k/e_1, e_2$  and their  $\mathbb{Z}$ -labellings; unlabelled edges have group-value 0.

Let  $P = \{a_1, \dots, a_k\} \cup \{d_1, \dots, d_k\}$  and  $Q = \{b_1, \dots, b_k\} \cup \{c_1, \dots, c_k\}$ . Note that  $P \cup \{e_1, e_2\}$  and  $Q \cup \{e_1, e_2\}$  are circuit-hyperplanes of  $N_k$ ; let  $M_k^L$  be the matroid obtained from  $N_k$  by relaxing  $P \cup \{e_1, e_2\}$  and  $Q \cup \{e_1, e_2\}$ .

We will prove that  $M_k^L/\{e_1, e_2\}$  is an excluded minor. We start by showing that proper minors of  $M_k^L/\{e_1, e_2\}$  are lifted-graphic matroids; this is almost a carbon copy of the proof of Lemma 3.1.

**Lemma 4.1.** *For each  $e \in P \cup Q$ , both  $M_k^L/e$  and  $M_k^L \setminus e$  are lifted-graphic matroids.*

*Proof.* Let  $M_P$  and  $M_Q$  denote the matroids obtained from  $M_k^L$  by tightening the free bases  $P \cup \{e_1, e_2\}$  and  $Q \cup \{e_1, e_2\}$  respectively. For

each  $e \in P$ , we have  $M_k^L \setminus e = M_P \setminus e$  and  $M_k^L / e = M_Q / e$ . Similarly, for each  $e \in Q$ , we have  $M_k^L / e = M_P / e$  and  $M_k^L \setminus e = M_Q \setminus e$ . So it suffices to prove that  $M_P$  and  $M_Q$  are frame matroids.

Note that  $M_P$  and  $M_Q$  are obtained from  $N_k$  by relaxing the circuit-hyperplanes  $Q \cup \{e_1, e_2\}$  and  $P \cup \{e_1, e_2\}$  respectively. Since  $G_k$  is a lifted-graphic representation of  $N_k$  and  $G_k[Q \cup \{e_1, e_2\}]$  is a cycle in  $G_k$ , we have that  $G_k$  is a lifted-graphic representation of  $M_P$ ; so  $M_P$  is indeed a lifted-graphic matroid. Let  $G'_k$  be the graph obtained from  $G_k \setminus \{e_1, e_2\}$  by adding  $e_1$  connecting  $s_1$  to  $t_2$  and adding  $e_2$  connecting  $s_2$  to  $t_1$ . It is straightforward to verify that  $G'_k$  is a lifted-graphic representation of  $N_k$  (since  $\{e_1, e_2\}$  is a series pair in  $N_k$ ). Finally, since  $G'_k[P \cup \{e_1, e_2\}]$  is a cycle in  $G'_k$ , we have that  $G'_k$  is a lifted-graphic representation of  $M_Q$ ; so  $M_Q$  is indeed a lifted-graphic matroid.  $\square$

Now it remains to show that  $M_k^L / \{e_1, e_2\}$  itself is not a lifted-graphic matroid.

**Lemma 4.2.**  *$M_k^L / \{e_1, e_2\}$  is not a lifted-graphic matroid.*

*Proof.* Assume to the contrary that  $H$  is a lifted-graphic representation of  $M_k^L / \{e_1, e_2\}$ . Let  $A_1 = \{a_1, b_1, a_k, c_k\}$ ,  $A_2 = \{c_1, d_1, b_k, d_k\}$ ,  $B_1 = \{a_1, b_1, b_k, d_k\}$ ,  $B_2 = \{c_1, d_1, a_k, c_k\}$ ,  $C_1 = \{a_1, b_1, c_1, d_1\}$ , and  $C_2 = \{a_k, b_k, c_k, d_k\}$ . Let  $G = G_k / \{e_1, e_2\}$ ; Figure 1 depicts  $G$  with a  $\mathbb{Z}$ -labelling encoding its balanced cycles with respect to  $N_k / \{e_1, e_2\}$ . From this  $\mathbb{Z}$ -labelled graph we see that:

- (i) each cocircuit in  $N_k / \{e_1, e_2\}$  (and hence also in  $M_k^L / \{e_1, e_2\}$ ) has size at least 4, and
- (ii) The only 4-element cocircuits of  $N_k / \{e_1, e_2\}$  (and hence also of  $M_k^L / \{e_1, e_2\}$ ) are the sets  $\delta_{G_k}(v)$  for  $v \in V(G_k) - \{s_1, s_2, t_1, t_2\}$  and the sets  $A_1, A_2, B_1, B_2, C_1$ , and  $C_2$ .

**4.2.1.**  *$H$  is a loopless connected 4-regular graph.*

*Subproof.* Since  $|E(H)| = |P \cup Q| = 2k = 2|V(H)|$ , we see that  $H$  has average degree 4. It follows from (i) that  $H$  is 4-regular and loopless. Finally, since  $N_k / \{e_1, e_2\}$  is connected and  $H$  is loopless,  $H$  is connected.  $\square$

We will call a set  $X \subseteq E(H)$  *vertical* if there exists  $v \in V(H)$  such that  $X = \delta_H(v)$ . Now each of the  $2k$  vertical sets is a 4-element cocircuit of  $M_k^L / \{e_1, e_2\}$  and each element in  $P \cup Q$  is in exactly two vertical sets. We have listed all of the 4-element cocircuits of  $M_k^L / \{e_1, e_2\}$  in (ii). Note that the elements in  $\{a_2, a_3, \dots, a_{k-1}\} \cup \{d_2, d_3, \dots, d_{k-1}\}$  are each in exactly two 4-element cocircuits. It follows that, for each  $v \in V(G_n) - \{s_1, s_2, t_1, t_2\}$ , the set  $\delta_{G_k}(v)$  is vertical. There are three

possibilities for the pair of remaining vertical sets, namely,  $(A_1, A_2)$ ,  $(B_1, B_2)$ , and  $(C_1, C_2)$ .

First suppose that  $C_1$  and  $C_2$  are both vertical. Then  $H[\{a_1, c_1, a_3, b_3\}]$  is the union of two edge-disjoint cycles. So  $\{a_1, c_1, a_3, b_3\}$  dependent in  $M_k^L/\{e_1, e_2\}$  and hence also in  $N_k/\{e_1, e_2\}$ . However, from the definition of  $N_k$ , the set  $\{e_1, e_2, a_1, c_1, a_3, b_3\}$  is independent. From this contradiction we have that the remaining pair of vertical sets is either  $(A_1, A_2)$  or  $(B_1, B_2)$ .

Now, since  $k$  is odd, one of  $H[P]$  and  $H[Q]$  is a cycle while the other is the union of two vertex-disjoint cycles. However  $P$  and  $Q$  are both independent in  $M_k^L/\{e_1, e_2\}$  which contradicts the fact that  $H$  is a lifted-graphic representation.  $\square$

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